→ **Conservation Laws in MHD**

- here discuss: conservation $\begin{cases}\text{momentum}\\ \text{energy}\\ \text{angular momentum}\end{cases}$

and virial theorems

> **Momentum** → key: construct evolution of momentum density

have:
$$\rho\left(\frac{\partial \underline{V}}{\partial t} + \underline{V}\cdot\nabla\underline{V}\right) = -\nabla\left(P+\frac{B^2}{8\pi}\right) + \frac{\underline{B}\cdot\nabla\underline{B}}{4\pi} + \underbrace{\rho\underline{g}}_{\text{body force}}$$

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\underline{V}) = 0$$

$$\Rightarrow \underbrace{\frac{\partial}{\partial t}(\rho\underline{V})}_{\substack{\text{momentum}\\ \text{density}}} + \nabla\cdot\underbrace{(\rho\underline{V}\,\underline{V})}_{\substack{\text{Reynolds stress}\\ \text{tensor}}} = -\nabla\left(P+\frac{B^2}{8\pi}\right) + \underbrace{\nabla\cdot\frac{\underline{B}\,\underline{B}}{4\pi}}_{\substack{\text{Maxwell stress}\\ \text{tensor}}} + \rho\underline{g}$$

$$\underline{T}_R = \rho\underline{V}\,\underline{V}$$

$$T_B = \frac{B^2}{8\pi}\underline{\underline{I}} - \frac{\underline{B}\,\underline{B}}{4\pi}$$

thus re-write:

$$\frac{\partial}{\partial t}(\rho\underline{V}) = -\nabla\cdot\underline{\underline{T}} + \rho\underline{g}$$

where

$$\underline{\underline{T}} = \left(p + \frac{B^2}{8\pi}\right)\underline{\underline{I}} + \frac{\underline{B}\,\underline{B}}{4\pi} - \rho\,\underline{V}\,\underline{V}$$

$$T_{ij} = \left(p + \frac{B^2}{8\pi}\right)\delta_{ij} + \frac{B_i B_j}{4\pi} - \rho V_i V_j$$

Then, if consider a 'blob' of {plasma / magnetofluid}:

surface S — crucial

$$\frac{\partial}{\partial t}\underbrace{\underline{P}}_{\text{momentum}} = \int d^3x\, \frac{\partial}{\partial t}\overbrace{(\rho\underline{V})}^{\text{momentum density}}$$

$$= -\int d^3x\, \underline{\nabla}\cdot\underline{\underline{T}} + \underbrace{\int d^3x\, \rho\underline{g}}_{\text{net body force}}$$

$$= -\int d\underline{S}\cdot\underline{\underline{T}} + \int d^3x\, \rho\underline{g}$$

So, apart from volume integrated body force,

$$\frac{\partial P}{\partial t} = -\int d\underline{S}\cdot\underline{\underline{T}}$$

{change in momentum set by stress on surface of blob}

$$\underline{\underline{T}} = \left(p + \frac{B^2}{8\pi}\right)\underline{\underline{I}} - \frac{\underline{B}\,\underline{B}}{4\pi} + \rho\,\underline{V}\,\underline{V}$$

Thus, can identify ways momentum is lost by the blob:

$- \underline{\underline{T}}_R \cdot d\underline{S} = -\rho \underline{V}\,\underline{V} \cdot d\underline{S}$ → flux of momentum density thru surface

$- \underline{\underline{T}}_{P_{tot}} \cdot dS = -\left(P + \frac{B^2}{8\pi}\right) \cdot d\underline{S}$ → pressure (total) force on surface, in $-d\underline{S}$ direction

$- \underline{\underline{T}}_{\substack{Mag\\ten}} \cdot d\underline{S} = \frac{\underline{B}\,\underline{B} \cdot d\underline{S}}{4\pi}$ → magnetic tension force in $+\underline{B}$ direction, piercing surface

$\sim (\underline{B} \cdot d\underline{S})\ \frac{\underline{B}}{4\pi}$ → tension of $\underline{B}/4\pi$, per line

$(\underline{B} \cdot d\underline{S})$ → # of lines thru $d\underline{S}$ outward

i.e

dS

→ Note that magnetic tension is independent of sign of $\underline{B}$ (as it should, tension is, strictly speaking, a dyad $\overleftrightarrow{S}$, not $\vec{P}$)

↳ tension field $\sim \underline{B}\,\underline{B}$

→ can make obvious analogy between strings, and field lines

$\longrightarrow B_0$

Alfven wave

$\#\text{strings}/\text{area} = B$

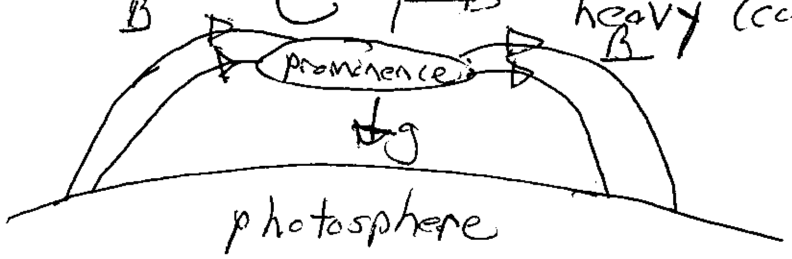
$\sigma = \rho/B$ → mass per length, of string

$T = B/4\pi$

$V_{ph}^2 = T/\sigma$
$= B^2/4\pi\rho_0$
$= V_A^2$

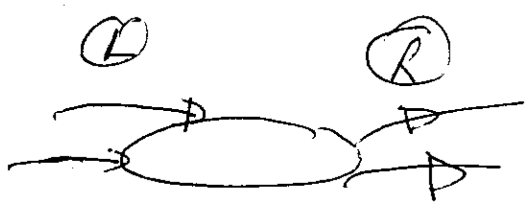
→ example: { Solar prominence (see cover of Kulsrud)

heavy (cool) → requires support against gravity



{ prominences often associated with radiative condensation

i.e.



L → #lines/area $= \underline{B}\cdot d\underline{S} < 0$ (inward)

force/Line is toward → lower right

∴ $\underline{F}_{L}$ → toward upper left

R → #lines/area $= \underline{B}\cdot d\underline{S} > 0$

f/Line is toward upper left

F_{R} → toward upper right

thus → prominence supported by magnetic tension (a la hammock — string)

→ squashing $\underline{B}$ → support by magnetic pressure, too ....

→ The Skeptic : " what of EM Momentum ?"

$$\underline{P}_{EM} = \underline{E} \times \underline{B} / 4\pi c$$

$$E \sim \frac{VB}{c} \Rightarrow P_{EM} \sim (\rho V)\, B^2/4\pi\rho c^2$$

$$\sim \rho V \left(V_A^2/c^2 \right) \ll 1$$

for $V_A \ll c$..

n.b. obviously important in relativistic and EMHD !

→ Angular Momentum → read Kulsrud ---

→ <u>Energy</u>

$$\text{Now energy}: E = \underset{\text{kinetic}}{E_V} + \underset{\text{thermal}}{E_p} + \underset{\text{magnetic}}{E_B} + \underset{\text{gravity}}{E_g}$$

$$E = \int_V d^3x \left[\frac{1}{2}\rho V^2 + \frac{P}{\gamma - 1} + \frac{B^2}{8\pi} + \frac{\rho\phi}{2} \right]$$

self-gravitating blob

$$\text{where} \begin{cases} \underline{g} = -\underline{\nabla}\phi \\ \nabla^2\phi = 4\pi G\rho \end{cases}$$

i.e. $\underline{g}$ evolves self-consistently (not "constant")

N.B. Problem # 11 : Jeans Instability

→ Calculate the growth rate of density perturbations in an un-magnetized, self-gravitating fluid

→ repeat in 1D using Vlasov equation

→ Where does E_p come from?

Consider work to compress plasma/fluid, i.e.

$$dW = -p\,dV$$

$$\Delta E = -\int_0^{\rho_0} P(\rho)\, d(1/\rho) = \int_0^{\rho_0} (\rho/\rho_0)^{\gamma} P_0 \frac{d\rho}{\rho^2}$$

$$= \frac{P_0}{\rho_0(\gamma-1)} \Rightarrow \underset{\substack{\uparrow \\ \text{energy} \\ \text{density}}}{\mathcal{E}} = \rho_0 \Delta E = \frac{P_0}{(\gamma-1)}$$

→ for energy balance, crank it out, using MHD equations ....

$$\frac{dE}{dt} = \underset{①}{\frac{dE_V}{dt}} + \underset{②}{\frac{d}{dt}E_p} + \underset{③}{\frac{d}{dt}E_B} + \underset{④}{\frac{d}{dt}E_g}$$

① $$\frac{d}{dt}E_V = \int d^3x \frac{\partial}{\partial t}\left(\frac{\rho V^2}{2}\right)$$

$$= \int d^3x \left[V^2 \frac{\partial \rho}{\partial t} + \rho \frac{\partial \underline{V}}{\partial t} \cdot \underline{V} \right] d^3x$$

↳ ibp leaves $\underline{S.T.}$ and cancels 2nd.

$$= \int d^3x \left[-\frac{\underline{V}^2}{2} \nabla\cdot(\rho \underline{V}) - \underline{V}\cdot\rho(\underline{V}\cdot\nabla\underline{V}) - \underline{V}\cdot\nabla p + \underline{V}\cdot(\underline{J}\times\underline{B}) - \rho\underline{V}\cdot\nabla\phi \right]$$

i.e. $\int -\frac{V^2}{2}\nabla\cdot(\rho \underline{V}) = -\frac{V^2}{2}\rho \underline{V}\Big| + \int(\underline{V}\cdot\nabla\underline{V})\cdot\rho\,\underline{V}$

cancels 2nd term in $\frac{dE_V}{dt}$

(2) $$\frac{d}{dt}E_p = \int \frac{d^3x}{\gamma-1}\frac{\partial p}{\partial t}$$

Now eqn. state $\Rightarrow$ $\frac{1}{p}\frac{dp}{dt} + \frac{\gamma}{\rho}\frac{d\rho}{dt} = 0$

and $\frac{1}{\rho}\frac{d\rho}{dt} = \nabla\cdot\underline{V}$ $\left[\frac{1}{(p/\rho^\gamma)}\frac{d}{dt}(p/\rho^\gamma) = 0\right]$

$$\Rightarrow \frac{\partial p}{\partial t} = -\underline{V}\cdot\nabla p - \gamma p\,\nabla\cdot\underline{V}$$

so $$\frac{d}{dt}E_p = \frac{-1}{(\gamma-1)}\int d^3x\,(\underline{V}\cdot\nabla p + \gamma p\,\nabla\cdot\underline{V})$$

$$= -\int d^3x\left[\frac{\gamma}{\gamma-1}\nabla\cdot(p\underline{V}) - \underline{V}\cdot\nabla p\right]$$

yields a surface term

cancels $\underline{V}\cdot\nabla p$ term in $\frac{dE_n}{dt}$

expect similar relation between $\underline{J}\times\underline{B}$ and $\frac{\partial}{\partial t}B^2$ .... so...

③ $$\frac{d}{dt} E_B = \frac{1}{4\pi}\int d^3x \ \underline{B}\cdot\frac{\partial \underline{B}}{\partial t}$$

$$= \frac{1}{4\pi}\int d^3x \ \underline{B}\cdot(\underline{\nabla}\times\underline{V}\times\underline{B})$$ by induction eqn.

ⓐ ⓑ

$$= -\int d^3x \left\{ \underbrace{\underline{\nabla}\cdot\left[\frac{\underline{B}\times(\underline{V}\times\underline{B})}{4\pi}\right]}_{\substack{\text{surface term} \\ (\to \text{Poynting})}} - \underbrace{\frac{(\underline{\nabla}\times\underline{B})}{4\pi}\cdot(\underline{V}\times\underline{B})}_{\underline{J}\cdot\underline{V}\times\underline{B}} \right\}$$

ⓑ $$= \int d^3x \ \underline{J}\cdot(\underline{V}\times\underline{B}) = -\int d^3x \ (\underline{J}\times\underline{B})\cdot\underline{V}$$

cancels $\underline{V}\cdot\underline{J}\times\underline{B}$ term in dE_V/dt

which leaves:

④ $$\frac{dE_g}{dt} = \frac{1}{2}\int d^3x \left(\phi\frac{\partial\rho}{\partial t} + \rho\frac{\partial\phi}{\partial t}\right)$$

$$= \frac{1}{2}\int d^3x \ \phi\frac{\partial\rho}{\partial t} + \int d^3x \frac{\nabla^2\phi}{8\pi G}\frac{\partial\phi}{\partial t}$$ ibp →

$$= \frac{1}{2}\int d^3x \ \phi\frac{\partial\rho}{\partial t} d^3x + \int \frac{\phi}{8\pi G}\nabla^2\frac{\partial\phi}{\partial t} d^3x$$

so
$$\frac{dE_g}{dt} = \frac{1}{2}\int \phi \frac{\partial \rho}{\partial t} d^3x + \frac{1}{2}\int d^3x\, \phi \frac{\partial \rho}{\partial t}$$

$$= \int d^3x\, \phi \frac{\partial \rho}{\partial t} = -\int d^3x\, \phi\, \nabla\cdot(\rho \underline{V})$$

$$= +\int d^3x\, \rho \underline{V}\cdot\underline{\nabla}\phi$$

cancels
$-\rho \underline{V}\cdot\nabla\phi$ in $\frac{dE_4}{dt}$ $\quad + -\int d\underline{S}\cdot \rho\phi \underline{V}$

Note: $-\underline{V}\cdot\underline{\nabla}P$; $\underline{V}\cdot(\underline{J}\times\underline{B})$; $-\rho\underline{V}\cdot\underline{\nabla}\phi$; $\underline{V}\cdot\rho\underline{V}\cdot\underline{\nabla}\underline{V}$

terms <u>all</u> <u>cancel</u> in dE_V/dt !

<u>Now adding up all 4 pieces</u> $\Rightarrow$

$$\boxed{\frac{d}{dt}E = -\int d\underline{S}\cdot\left[\rho\underline{V}\frac{V^2}{2} + \frac{\gamma}{\gamma-1}p\underline{V} - \frac{(\underline{V}\times\underline{B})\times\underline{B}}{4\pi} + \rho\underline{V}\phi\right]}$$

i.e. not surprisingly, only survivors are surface terms . . . . $\Rightarrow$ in ideal MHD, only change in energy of blob involves boundary . . .

i.e. have:

$$\frac{dE}{dt} = -\int d\underline{S} \cdot \left[\overset{①}{\rho \underline{V} \frac{V^2}{2}} + \overset{②}{\frac{\gamma}{\gamma-1} p \underline{V}} - \overset{③}{\frac{(\underline{V}\times\underline{B})\times\underline{B}}{4\pi}} + \overset{④}{\rho \underline{V} \phi} \right]$$

① → kinetic energy loss via simple kinetic energy flow thru surface.

② → $-\frac{\gamma \underline{V} \cdot d\underline{S}\, p}{\gamma-1}$ ⟷ outward flow of enthalpy

i.e. $-\frac{\gamma\, p \underline{V} \cdot d\underline{S}}{\gamma-1} = -\frac{p}{\gamma-1} \underline{V} \cdot d\underline{S} - p\, \underline{V} \cdot d\underline{S}$

$-\frac{p}{\gamma-1} \underline{V} \cdot d\underline{S}$ → outward flow of thermal energy $\left(d\underline{S} \cdot \underline{V} \frac{p}{\gamma-1} \right)$ thru S

$-p\, \underline{V} \cdot d\underline{S}$ → $p\,dV$ work of blob on exterior

③

as $\underline{E} = -\frac{\underline{V}\times\underline{B}}{c}$ . . . →

so ③ $= d\underline{S} \cdot \frac{\underline{E}\times\underline{B}}{4\pi c}$ → $\left\{ \begin{array}{l} \text{loss of} \\ \text{energy by} \\ \text{Poynting flux} \end{array} \right.$

④ loss of gravitational potential energy due outflow from blob . . .

It's all clear!!

this brings us to ....

→ <u>Virial Theorems in MHD</u>

- what is a virial theorem

- why yet another theorem?

→ Virial Theorems are:

- space/time averaged energy theorems
- "lumped parameter" relations for energies in complex, multi-element interacting systems
- useful for 'back-of-envelope' estimates, etc.
- logically extend the moment program:

$$\underbrace{f(\underline{x},\underline{v},t)}_{\substack{\text{phase space}\\ \text{fluid}}} \xrightarrow[\underline{v}\text{ moments}]{} \underbrace{n(\underline{x},t),\ \underline{V},\ T}_{\substack{\text{position space}\\ \text{fluid}}} \xrightarrow[\substack{\text{virial}\\ \text{integrals}}]{} \underbrace{E_v,\ E_B,\ \text{etc.}}_{\#}$$

Before proceeding:

¿ Can an <u>isolated</u> blob of MHD plasma <u>confine</u> ... <u>itself</u> without self gravity ?

Easily answered by Virial Theorem ....

Recall, for system of particles, Virial theorem derived by considering:

$$\frac{d}{dt}\left(\sum_i \underline{p}_i \cdot \underline{x}_i\right) = \sum_i \underline{p}_i \cdot \dot{\underline{x}}_i + \sum_i \dot{\underline{p}}_i \cdot \underline{x}_i$$

$$= \underbrace{2T}_{\text{kinetic energy}} + \underbrace{\sum_i \left(-\frac{\partial U}{\partial \underline{x}_i}\right) \cdot \underline{x}_i}_{\text{via Newton's Law}}$$

Now, if $\sum_i \underline{p}_i \cdot \underline{x}_i$ bounded,

$$\left\langle \sum_i \underline{p}_i \cdot \underline{x}_i \right\rangle = \frac{1}{T}\int_0^T dt \sum_i \underline{p}_i \cdot \underline{x}_i$$

$$\underset{T \to \infty}{\longrightarrow} 0$$

so ...

$$2\langle T\rangle = \Big\langle \underbrace{\sum_i \frac{\partial U}{\partial \underline{x}_i}\cdot \underline{x}_i}_{\text{(First) Virial of system}} \Big\rangle$$

Further, if $U = U(\underline{x}_1, \underline{x}_2, \cdots \underline{x}_n)$

where $U(\alpha\underline{x}_1, \alpha\underline{x}_2, \ldots \alpha\underline{x}_n) = \alpha^k U(\underline{x}_1, \underline{x}_2, \cdots \underline{x}_n)$ (scaling $\Leftrightarrow$ structure of power-law potentials $\rightarrow$ i.e. H.O. $\rightarrow k=2$, Coulomb $\rightarrow k=-1$)

$\rightarrow$ homogeneous function

$\Rightarrow$

$$2\langle T\rangle = k\langle U\rangle$$

but of course:

$$T + U = \langle T\rangle + \langle U\rangle = E$$

then

$$\left(\frac{k}{2}+1\right)\langle U\rangle = E$$

$$\langle U\rangle = \frac{2}{k+2}E \quad , \quad \langle T\rangle = \frac{kE}{k+2}$$

check: $k=2$, $\langle U\rangle = \frac{1}{2}E$, $\langle T\rangle = \frac{1}{2}E$ ✓

$k=-1$, $\langle T\rangle = -E$ ✓ $\Rightarrow$ (bounded motion only if total energy negative (i.e. bound state))

$(\Rightarrow E<0)$

Aside: Simplest realization of negative specific heat 'paradox', i.e.

 → consider 'blob' of self gravitating matter

$$E \sim -1/R$$

if radiation  → E decreases ⇔ R decreases

$$\therefore \underbrace{(-E)}_{\text{kinetic energy}} \text{ increases} \Rightarrow \langle T \rangle \text{ increases}$$

but $\langle T \rangle \sim$ temperature, so have cycle of: radiative cooling ⇒ temperature increase?!

$$\Rightarrow \underbrace{c}_{\text{specific heat}} < 0 \;!?$$

In the days before the discovery of nuclear fusion, this was thought to be what heated stars. Kelvin, in particular, was a proponent.

Now, proceeding to full Virial theorem ...

- Consider equation of motion

$$\underbrace{\frac{\partial}{\partial t}(\rho v_i)}_{\text{momentum}} = -\frac{\partial}{\partial x_j}\underbrace{T_{ij}}_{\underline{\text{full}}\text{ stress tensor}}$$

$$T_{ij} = \rho v_i v_j + \left(p + \frac{B^2}{8\pi}\right)\delta_{ij} - \frac{B_i B_j}{4\pi} + \cancel{\rho\phi\delta_{ij}}$$

Now, recalling relation of virial to $\frac{d}{dt}(\underline{p}\cdot\underline{x})$
$\Rightarrow$ consider:

$$I_{ij} = \int d^3x\, \rho\, x_i x_j \qquad (\sim \text{moment of inertia})$$

$\mapsto$ Virial theorem is for $\underline{\text{tensor}}$ ---

and
$$\frac{d}{dt} I_{ij} = \int d^3x\, \frac{\partial\rho}{\partial t} x_i x_j$$

$$= -\int d^3x\, \frac{\partial}{\partial x_t}(\rho v_t)\, x_i x_j$$

integrating by parts assuming ρ compact (i.e. 'blob' of interest)

$$= \int d^3x\, [\rho x_i v_j + \rho x_j v_i]$$

$\underline{\text{so}}$

$$\frac{d^2 I_{ij}}{dt^2} = \int d^3x \left[x_i\left(\frac{\partial}{\partial t}\rho v_j\right) + x_j \frac{\partial}{\partial t}(\rho v_i)\right]$$

but $\quad \frac{\partial}{\partial t}(\rho V_i) = -\frac{\partial}{\partial x_k} T_{ik}$

$\Rightarrow$

$$\frac{d^2 I_{ij}}{dt^2} = -\int d^3x \left[x_i \frac{\partial T_{j,t}}{\partial x_t} + x_j \frac{\partial T_{i,t}}{\partial x_t} \right]$$

and integrating by parts, assuming $\left\{ \begin{array}{l} \text{compact blob,} \\ \text{no external} \\ \text{linkage} \end{array} \right.$

$\Rightarrow$

$$\frac{d^2 I_{ij}}{dt^2} = +\int d^3x \left[\delta_{i,t} T_{j,t} + \delta_{j,t} T_{i,t} \right]$$

$\partial x_i / \partial x_t = 0$
unless $i = t$

$$= +\int d^3x \left[T_{j,i} + T_{i,j} \right]$$

and as T_{ij} manifestly symmetric $\Rightarrow$

$$\boxed{\begin{array}{l} \frac{1}{2} \frac{d^2 I_{ij}}{dt^2} = +\int d^3x \, T_{ij} \\ T_{ij} = \rho v_i v_j + \left(p + \frac{B^2}{8\pi}\right) \delta_{ij} - \frac{B_i B_j}{4\pi} + \rho \phi \delta_{ij} \end{array}}$$

— tensor virial theorem.

Now, to make contact with notions of energy, etc., useful to contract the tensor

$$I = I_{ii} = \mathrm{tr}\, I_{ij}$$

repeated indexes summed

tr (V.T.) $\Rightarrow$

$$\mathrm{tr}\ \frac{1}{2}\frac{d^2 I_{ij}}{dt^2} = \frac{d^2}{dt^2}\left(\int d^3x\ \frac{\rho x^2}{2}\right)$$

$$= \mathrm{tr}\int d^3x \left[\rho v_i v_j + \left(p + \frac{B^2}{8\pi}\right)\delta_{ij} - \frac{B_i B_j}{4\pi} + \rho\phi\,\delta_{ij}\right]$$

$$= \int d^3x \left[\rho v^2 + 3\left(p + \frac{B^2}{8\pi}\right) - \frac{B^2}{4\pi} + 3\rho\phi\right]$$

$\therefore\ I \equiv \int d^3x\ \rho x^2/2 \ \Rightarrow$

$$\boxed{\frac{d^2 I}{dt^2} = \int d^3x \left[\rho v^2 + 3p + \frac{B^2}{8\pi} + 3\rho\phi\right]}$$

$\rightarrow$ Scalar Virial Theorem.

Now, first neglect self-gravitation $\Rightarrow$

$$\frac{d^2 I}{dt^2} = \frac{d^2}{dt^2}\left(\int d^3x \, \frac{\rho x^2}{2}\right)$$

$$= \int d^3x \left[\rho V^2 + 3p + B^2/8\pi\right]$$

Now $\rightarrow$ can an isolated blob of MHD fluid confine itself??

If 'self-confined' $\Rightarrow$ $\frac{dI}{dt} \leq 0$

$\frac{d^2 I}{dt^2} \leq 0$

i.e. quiescent $\Rightarrow \dot{I}, \ddot{I} = 0$

stable pulsation $\Rightarrow \ddot{I} = -\Omega^2 I < 0$

but have $\ddot{I} = \int d^3x \left[\rho V^2 + 3p + B^2/8\pi\right]$

so even if $V^2 = 0$ (no fluid motion in blob) $\Rightarrow$

$p > 0$, $B^2/8\pi > 0$ $\Rightarrow$ $\ddot{I} > 0$!

$\therefore$ No $\rightarrow$ isolated blob can't confine itself.

More generally, noting that

$$E_V = \frac{1}{2}\int d^3x\, \rho V^2/2$$

$$E_p = \int d^3x\, \frac{P}{\gamma-1} = \frac{3}{2}\int d^3x\, P \qquad \text{(gas)}$$

$$E_B = \int d^3x\, \frac{B^2}{8\pi}$$

can write scalar Virial theorem in form:

$$\boxed{\frac{d^2I}{dt^2} = 2E_V + 2E_p + E_B}$$

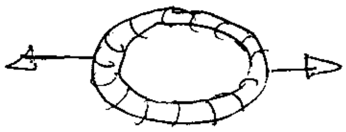
simple relation in terms energies

Aside: $\Rightarrow$ So, isolated blob can't confine itself

$\Rightarrow$ how is $\left\{\begin{array}{l}\text{tokamak} \rightarrow B_T \text{ for } \left\{\begin{array}{l}\text{stability; not} \\ \text{transport}\end{array}\right. \\ \underline{\text{or}} \text{ - better} \quad \text{macro - confinement} \\ \text{RFP} \rightarrow \text{weak external } B_T \text{ guide} \\ \text{(negligible)}\end{array}\right.$

confined ?? Confinement by wall is unacceptable ...

Answer: → toroidal plasma tends to expand toroidally



→ held in place by { conducting shell (often undesireable) or "vertical field"

i.e.

→ additional external P_{Mag} to oppose toroidal expansion — vertical field

→ image currents in close-in conducting shell can do likewise

JET anecdote re: vertical field failure ...

Now, retaining self-gravitation:

$$T_{ij}\Big|_{\text{gravity}} = \rho\phi\,\delta_{ij} = 2\underbrace{\left(\frac{\rho\phi}{2}\right)}_{\mathcal{E}_{\text{gravity}}}\delta_{ij}$$

to calculate: $\nabla^2\phi = 4\pi G\rho$

$$\Rightarrow \quad \phi = -G\int d^3x\,\frac{\rho(x')}{|x-x'|}$$

so

$$T_{ij}\Big|_{\text{gravity}} = T\Big|_{\text{gravity}} \delta_{ij}$$

$\Rightarrow$

$$T = -\frac{G}{2}\int d^3x \int d^3x' \, \frac{\rho(\underline{x})\,\rho(\underline{x}')}{|\underline{x}-\underline{x}'|}$$

$$= + E_{\text{gravitation}} \equiv E_g < 0$$

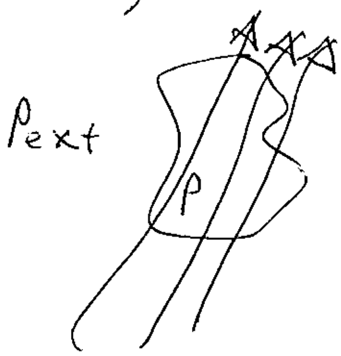
so scalar Virial theorem becomes, with gravity $\Rightarrow$

$$\boxed{\frac{1}{2}\frac{d^2 I}{dt^2} = 2E_v + 2E_p - |E_g|}$$

so with gravity, can have self-confining blob
(no surprise ...)

This brings us to another application of Virial theorems, name proto-stellar cloud collapse ....

— now, consider a plasma cloud/blob



- mass M, radius R
- "threaded" by B
- pressure p, external pressure P_0
- no bulk motion
- frozen flux

now, easy to show for $\ddot{I} = 0$, $\underline{V} = 0$, must have:

surface terms

$$2E_p - |E_g| + E_B = \int dA\, P_{ext}\, \hat{x}\cdot\hat{n} - \int dA\, \underline{x}\cdot\underline{\underline{T}}_B\cdot\hat{n}$$

external pressure.

magnetic stress thru surface.

Now, can estimate:

$M = \int \rho\, dV \rightarrow$ total mass

$E_p \cong C_s^2 M$

$|E_g| \cong \underbrace{\alpha}_{\text{form factor}} \dfrac{GM^2}{R}$

For frozen flux, $\Phi \sim \pi R^2 B$

so $\quad E_B + \int dA\, \underline{x} \cdot \underline{\underline{T}}_B \cdot \hat{n} \sim \beta \Phi^2 / R$

⇒ have: (eliminating extraneous factors):

$$\boxed{R^2 P_{ext} \sim \left(\frac{\beta \Phi^2}{R} - \alpha \frac{GM^2}{R} + \frac{3}{2} c_s^2 M \right)}$$

→ scalar virial theorem for cloud...

Now: $\quad P_{ext} \sim \left(\frac{\beta \Phi^2}{R^3} - \alpha \frac{GM^2}{R^3} + \frac{3}{2} \frac{c_s^2 M}{R^2} \right)$

→ if $\Phi, G \to 0$ → need $P_{int} = P_{ext}$ for confinement

⇒ if $\Phi = 0$

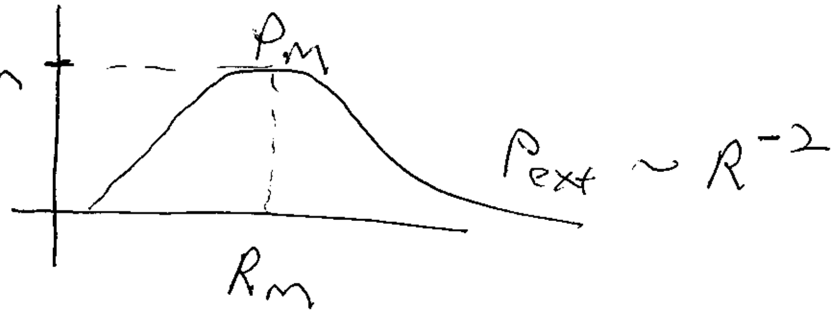
$$P_{ext} = -\frac{\alpha G M^2}{R^3} + \frac{3}{2} \frac{c_s^2 M}{R^2}$$

$dP/dR = 0 \quad ⇒ \quad 3\alpha \frac{GM^2}{R^4} = 3 \frac{c_s^2 M}{R^3}$

$$R_{max} = GM\alpha / c_s^2$$

[Note: ⇒ $R_m^2 = (G\rho / c_s^2)^{-1/2}$ ⇒ L^2_{Jeans}]

so P_{ext} needed



- $P > P_{max} \rightarrow$ no equilibrium

- $R < R_{max} \rightarrow P_{ext}$ must decrease to maintain equilibrium $\Rightarrow$ instability to gravitational collapse

$\Rightarrow \Phi \neq 0$ (magnetic field on ...) $\rightarrow$ note immediately that magnetic support scales similarly to gravitational attraction

$\Rightarrow$

$$P_{ext} \sim \left[\left(\beta \Phi^2 - \alpha G M^2\right)/R^3 + \frac{3}{2} \frac{C_s^2 M}{R^2} \right]$$

so key point is: $\left(\beta \Phi^2 - \alpha G M^2\right) \lessgtr 0$

$$\Rightarrow M \gtrless M_\Phi = \sqrt{\beta/\alpha}\, \Phi / G^{1/2}$$

i.e

$M < M_\Phi \rightarrow$ magnetically subcritical mass for gravitational collapse

$M > M_\Phi$ → magnetically super-critical mass for collapse!

i.e. $M < M_\Phi$ → repulsive effects $\left\{\begin{array}{l}\text{field}\\ \text{thermal}\end{array}\right\}$ pressure always win
$(M_\Phi^2 - M^2 > 0)$
→ no ammount of external compression can induce indefinite contraction, IF flux remains frozen in

$M > M_\Phi$ → sufficient external pressure/ compression can induce gravitational collapse, even if flux frozen in.
$(M_\Phi^2 - M^2 < 0)$

[Note: If kinetic energy contribution, NL Alfven waves can support cloud.]

For perspective, recall:

- (famous) Chandrasekhar Mass
  - $M > M_{Chandra}$ → collapse
  - $M < M_{Chandra}$ → no collapse

$M_{Chandra}$ derived for degenerate Fermi gas equation of state → $\gamma = 4/3$, instead of $\gamma = 5/3$.

– of flux-freezing $\Rightarrow$ $\Phi \sim B R^2$

$\rho R^3 \sim M$

$\Rightarrow B \sim R^{-2}$ $\Rightarrow$ $B \sim \rho^{2/3}$

$\therefore$ $B^2 \sim P_{Mag} \sim \rho^{4/3}$

$\Rightarrow$ if flux frozen, field obeys equation of state like Fermi gas

(i.e. flux freezing is akin to exclusion, albeit on field-lines-per-fluid-element)

$\therefore$ an analogue to Chandrasekhar mass seems quite plausible ....

Aside: Chandrasekhar Limit — Simple Derivation
(c.f.: Shapiro, Teukolsky)

→ suppose: N Fermions in star of radius R

$\therefore \quad n_{\text{Fermion}} \sim N/R^3$

$\therefore \quad \text{Vol./Fermion} \sim 1/n$ (Pauli exclusion)

$\underbrace{p}_{\text{Fermion Momentum}} \sim \hbar/\Delta x \sim \hbar\, n^{1/3}$ (Heisenberg Uncertainty)

⇒ Fermion energy (per Fermion) : $E_F \simeq pc \sim \hbar c \dfrac{N^{1/3}}{R}$

replaces: (i.e. Thermal energy)

Gravitational Energy (per Fermion) : $E_{\text{Grav}} \sim -\dfrac{GM}{R} m_b$ → Baryon mass

$M \sim N\, m_B$

Pressure → electron
Mass → Baryon

$\therefore \quad E = E_F + E_G$

$$= \frac{\hbar c N^{1/3}}{R} - \frac{G N m_B^2}{R}$$

note: $E = E_F + E_G$

$$= \frac{\hbar c N^{1/3}}{R} - \frac{G N m_B^2}{R}$$

$E > 0 \Rightarrow$ decrease E, E_F by increasing R.

but as $E_F \downarrow$, electrons non-relativistic,
$\therefore E_F \sim 1/R^2 \rightarrow$ eqbrm.

$E < 0 \Rightarrow$ decrease E without bound by decreasing $R \Rightarrow$ collapse

$\therefore$ eqbrm: $\hbar c N^{1/3} = G N m_B^2$

$$N_{Max} = \left(\hbar c / G m_B^2\right)^{3/2} \sim 2 \times 10^{57} \quad \text{(proton)}$$

$$\therefore \quad M_{Chandra} = N_{max} m_B \sim 1.5 M_\odot$$

→ Magnetic Helicity

– another conserved quantity in ideal MHD is magnetic helicity K

$$K = \int_V d^3x\, \underline{A}\cdot\underline{B}$$

V is taken to be the volume of a 'flux tube'.

– what, yet another invariant!?!

→ K is different ⇒ has topological interpretation

$$K = \int_V d^3x\, A\cdot\underline{\nabla}\times\underline{A}$$

→ $\underline{x} \to -\underline{x}$ flips sign of K

→ K is a pseudo-scalar

∴ has orientation or "handedness"...

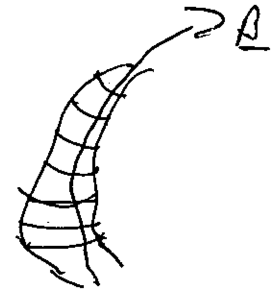
Proceed via:

– show K conservation
– discuss interpretation of K
– comment on utility ⇒ Taylor Relaxation

N.B.: Important → K is gauge invariant

i.e. if $\underline{A} \to \underline{A} + \underline{\nabla}\chi$

$$K \to K + \int_V d^3x\, \underline{\nabla}\chi \cdot \underline{B}$$



$$= K + \int_V d^3x\, \underline{\nabla} \cdot (\underline{B}\, \chi)$$

$= 0$, to surface term. $\left\{ \underline{B} \cdot \hat{n} = 0 \right.$ on surface of tube

Now, consider a blob of MHD fluid in motion

can show $\dfrac{dK}{dt} =$

$$\underline{E} + \frac{\underline{V} \times \underline{B}}{c} = \eta \underline{J}$$

$$\underline{E} = -\frac{1}{c}\frac{\partial \underline{A}}{\partial t} - \underline{\nabla}\phi$$

$\Rightarrow$

$$\frac{\partial \underline{A}}{\partial t} = \underline{V} \times \underline{\nabla} \times \underline{A} - c\underline{\nabla}\phi - c\eta \underline{J}$$

$$\frac{\partial \underline{B}}{\partial t} = -\underline{V} \cdot \underline{\nabla}\, \underline{B} + \underline{B} \cdot \underline{\nabla}\, \underline{V} - \underline{B}\, \underline{\nabla} \cdot \underline{V} + \eta \nabla^2 \underline{B}$$

$$\frac{dK}{dt} = \frac{d}{dt} \int_V d^3x\, (\underline{A} \cdot \underline{B})$$

$$= \int d^3x \left(\frac{d\underline{A}}{dt} \cdot \underline{B} + \underline{A} \cdot \frac{d\underline{B}}{dt} \right) + \int \underline{A} \cdot \underline{B}\, \frac{d}{dt}\, d^3x$$

$$\frac{dK}{dt} = \int d^3x \left(\frac{\partial \underline{A}}{\partial t} \cdot \underline{B} + (\underline{V} \cdot \nabla \underline{A}) \cdot \underline{B} + \underline{A} \cdot \frac{\partial \underline{B}}{\partial t} \right.$$

$$\left. + \underline{A} \cdot (\underline{V} \cdot \nabla \underline{B})) + \underline{A} \cdot \underline{B} \; \nabla \cdot \underline{V} \right)$$

where $\frac{d}{dt} d^3x = \nabla \cdot \underline{V}$

i.e. $\frac{d}{dt} dV = \frac{d}{dt} d\underline{\sigma} \cdot d\underline{\ell} + d\underline{\sigma} \cdot \frac{d}{dt} d\underline{\ell}$

$= - d\underline{\ell} \cdot \nabla \underline{V} \cdot d\underline{\sigma} + (\nabla \cdot \underline{V})(d\underline{\sigma} \cdot d\underline{\ell}) + d\underline{\ell} \cdot \nabla \underline{V} \cdot d\underline{\sigma}$

$= \nabla \cdot \underline{V} \; d^3\cancel{x}$ s.t. and $\underline{B} \cdot \hat{n}$ on surface of tube.

$$\frac{dK}{dt} = \int d^3x \left[\left(\underline{B} \cdot \cancel{\underline{V}} \times \underline{B} - c \cancel{\underline{B}} \cdot \nabla \phi - c\eta \, \underline{J} \cdot \underline{B} \right) \right.$$

$$\left. + \underline{A} \cdot \left(\nabla \times (\underline{V} \times \underline{B}) \right) + \nabla \cdot ((\underline{A} \cdot \underline{B}) \underline{V}) + \underline{A} \cdot \eta \nabla^2 \underline{B} \right]$$

where $\underline{A} \cdot (\underline{V} \cdot \nabla \underline{B}) + \underline{B} \cdot (\underline{V} \cdot \nabla \underline{A}) + \underline{A} \cdot \underline{B} \; \nabla \cdot \underline{V} = \nabla \cdot (\underline{V} \, \underline{A} \cdot \underline{B})$

$$\frac{dK}{dt} = \int d^3x \left[\nabla \cdot ((\underline{A} \cdot \underline{B}) \underline{V}) + \nabla \cdot ((\underline{V} \times \underline{B}) \times \underline{A}) + \cancel{(\underline{V} \times \underline{B})} \cdot (\nabla \times \underline{A}) \right.$$

$$\left. - c\eta \, \underline{J} \cdot \underline{B} - \eta (\underline{A} \cdot \nabla \times \underline{J}) \, c \right]$$

$$\Rightarrow \frac{dK}{dt} = \int d^3x \left\{ \underline{\nabla}\cdot\left[(\underline{A}\cdot\underline{B})\underline{v} + (\underline{v}\times\underline{B})\times\underline{A} + c\eta(\underline{A}\times\underline{J})\right] - c\eta\,\underline{J}\cdot\underline{B} - c\eta\,\underline{J}\cdot\underline{B}\right\}$$

$$= \int d\underline{S}\cdot\left[(\underline{A}\cdot\underline{B})\underline{v} + (\underline{v}\times\underline{B})\times\underline{A} + c\eta\,\underline{A}\times\underline{J}\right]$$

$$- 2\int d^3x\left[c\eta\,\underline{J}\cdot\underline{B}\right]$$

$$= \int d\underline{S}\cdot\left[\cancel{(\underline{A}\cdot\underline{B})\underline{v}} - \cancel{(\underline{A}\cdot\underline{B})\underline{v}} + \underbrace{(\underline{A}\cdot\underline{v})\underline{B}}_{0}\right] - c\eta\int d\underline{S}\cdot\underline{J}\times\underline{A}$$

$\underline{B}\cdot\hat{n} = 0$, on tube

$$- 2c\eta\int d^3x(\underline{J}\cdot\underline{B})$$

$$= -\int c\eta\, d\underline{S}\cdot\left[\underline{\nabla}\underline{B}\cdot\underline{A} - \underbrace{\cancel{\underline{A}\cdot\underline{\nabla}\underline{B}}}_{0}\right] - 2c\eta\int d^3x\,\underline{J}\cdot\underline{B}$$

$$= -2c\eta\int d^3x(\underline{J}\cdot\underline{B})$$

$\Rightarrow$ have shown:

$$\boxed{\frac{dK}{dt} = -2c\eta\int d^3x(\underline{J}\cdot\underline{B})}$$

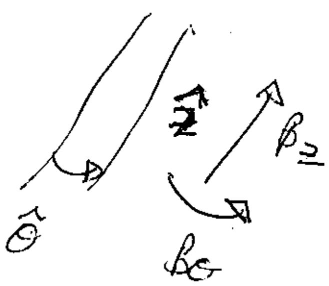
and clearly! $\frac{dK}{dt} \to 0$ as $\eta \to 0$ (non-singular $\underline{J}$)

$\therefore$ <u>Helicity</u> <u>is</u> <u>conserved</u> <u>in</u> <u>ideal</u> <u>MHD</u>.
<u>Magnetic</u>

$\Rightarrow$ Magnetic Helicity conserved, but what does it mean?

- helicity is non-trivial $\Rightarrow$ more than just helical field lines.

interesting to note: $q(r) = \frac{r B_z}{R B_\theta(r)} = \frac{1}{R\mu(r)}$



$\mu(r) = \frac{B_\theta(r)}{r B_z} \Rightarrow$ field line pitch.

cylindrical plasma $\Rightarrow \underline{B} = \underline{B}(r)$

Now, $A_\theta = \frac{1}{r}\int_0^r r B_z \, dr$

$$A_z = -\int_0^r B_\theta \, dr$$

so $$\underline{A}\cdot\underline{B} = \frac{B_\theta}{r}\int_0^r r B_z\,dr - B_z\int_0^r B_\theta\,dr$$

$$= \mu B_z \int_0^r \frac{B_\theta}{\mu}\,dr - B_z\int_0^r B_\theta\,dr$$

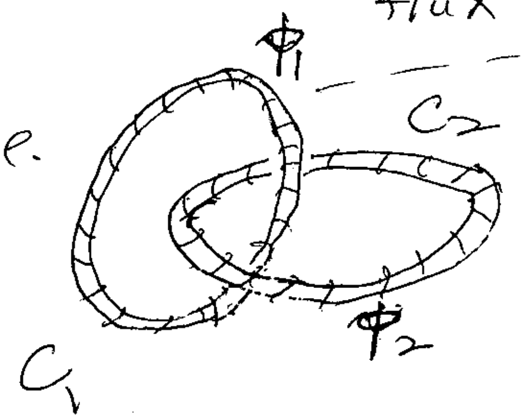
$$\underline{A}\cdot\underline{B} = B_z\left[\mu\int_0^r \frac{B_\theta}{\mu}\,dr - \int_0^r B_\theta\,dr\right]$$

$= 0$ for constant μ

$\therefore$ non-zero helicity requires $\mu = \mu(r)$
i.e. — pitch varies with radius
$\Rightarrow$ magnetic shear

— physically $\rightarrow$ helicity means self-linkage of flux tubes

i.e.



tube 1: flux
$$\Phi = \int d\underline{A}\cdot\underline{B} = \phi_1$$
x-section area — const

tube 2: $\Phi = \phi_2$

field in loops, only

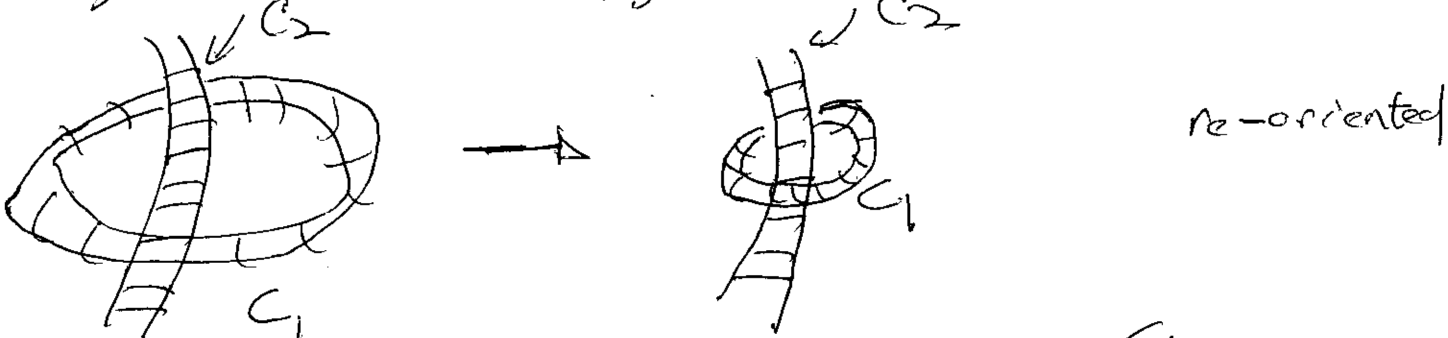
Now, for volume V_1 of tube 1

$$K = \int_{V_1} \underline{A}\cdot\underline{B}\, d^3x = \underbrace{\oint_{C_1}}_{\text{along loop}} d\ell \underbrace{\int_{A_1}}_{\text{x-sect area}} \underline{dS}\; A\cdot\underline{B}$$

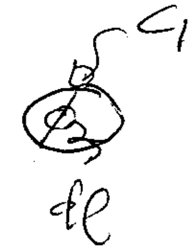
$$= \oint_{C_1} \underline{A}\cdot d\underline{\ell} \int_{S_1} \underline{B}\cdot\hat{n}\, dA$$

$$= \Phi_1 \oint_{C_1} \underline{A}\cdot d\underline{\ell}$$

Now, can shrink C_1, as no field outside loops



→ in x section:



but $\int_{C_1} A\cdot d\ell = \int_{A \text{ enclosed}} B\cdot dS = \Phi_2$

so . . . $K_1 = \Phi_1 \Phi_2$ → product of fluxes

similarly $K_2 = \Phi_2 \Phi_1$

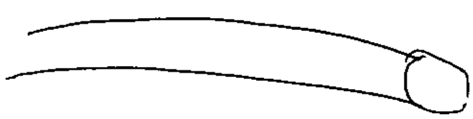
$\therefore \quad K = 2\Phi_1 \Phi_2$

if n windings $\quad K = K_1 + K_2 = \pm 2n\Phi_1 \Phi_2$.

⟹ helicity is measure of self-linkage of magnetic configuration.

Why care → Taylor Conjecture (1974) (J.B. Taylor)

- in magnetic confinement, of great interest to determine how fields, currents self-organize

- RFP  ~ toroid
 ~ toroidal current

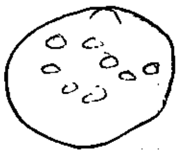
well fit by
$$B_z = B_0 J_0(\alpha r)$$
$$B_\theta = B_0 J_1(\alpha r)$$

$\underline{J} \times \underline{B} = 0$ ⟹ force free

⟹ why so robust ? especially since RFP so turbulent

- Taylor conjectured conservation of magnetic helicity constrains relaxation to force-free state.

Key Point
- helicity conserved in flux tubes, to η
- toroidal plasma $\rightarrow$ many small tubes



etc.

- recall Sweet-Parker model: magnetic reconnection / resistive dissipation effective on small scales.

$\Rightarrow$ Taylor Conjecture: At finite η, helicity of small tubes dissipated but, global helicity conserved.

i.e.

$$\int_{\text{plasma volume}} \underline{A}\cdot\underline{B}\, d^3x = K_0 \rightarrow \text{conserved.}$$

$\therefore$ Taylor conjectured that actual magnetic configuration could be explained by minimum principle:

$$\delta\left[\int_V d^3x \, \frac{B^2}{8\pi} + \lambda \int_V d^3x \, \underline{A}\cdot\underline{B}\right] = 0$$

i.e. minimize magnetic energy subject to constraint of conserved global helicity,

Comments:

→ it works! - indeed amazingly well - for RFPs, spheromaks, etc. Departures only recently being discovered

→ inspired idea of helicity injection as way to maintain configurations

→ it is a conjecture → no proof.

Hypothesis: Selective Decay → energy cascades → small scale; → helicity cascades → large scale (less dissipation)

→ Relevance to driven system?
i.e. in real RFP, transformer on.

→ dynamics? – <u>how</u> does relaxation occur

→ more in discussion of kinks, tearing..